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On Cantor-Bernstein type theorems in Riesz spaces

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ABSTRACT

We generalize the main result of [21] to Riesz spaces. Let X and Y be Riesz spaces with σ-complete Boolean algebras of projection bands. If X and Y are each Riesz isomorphic to a projection band of the other space then the spaces are Riesz isomorphic. As an application of the above theorem we give an example of non-Riesz isomorphic Banach lattices such that: (1) their order (= topological) duals are Riesz isomorphic and (2) each of them is Riesz isomorphic to a projection band of the other one.

1. INTRODUCTION

The following problem is fundamental in the theory of isomorphisms of topological vector spaces:

(P_1) *Let X and Y be tv spaces, f and g isomorphisms mapping X into Y and Y into X, respectively. Is X isomorphic to Y?*

L. Drewnowski has recently proved in [4] that if X and Y are nonseparable Banach spaces with symmetric uncountable Schauder bases then (P_1) has a positive answer. The answer is negative in general – even if we assume that X and Y are Banach spaces and f, g are isometries. A suitable example was given by S. Banach and S. Mazur [2]: $X=C(0,1)$ and $Y=C(0,1)\times l_1$ (see also [3], [12]). A. Pełzyński proved in 1960 that if $f(X)$ and $g(Y)$ are complemented subspaces of Y and X, respectively, and X, Y are isomorphic to their Cartesian squares, then the answer to question (P_1) is 'yes' (the so called Pełczyński's decomposition method; see also [8]). The condition $X\approx X\times X$ holds in classical

Banach spaces but it does not hold in general (see [3], [5]). Therefore the following problem is still open:

(P_2) *Let Banach (or F-)spaces X and Y be each isomorphic to a complemented subspace of the other space. Is X isomorphic to Y?*

In this paper we investigate analogous problems in Riesz spaces:

(P_3)[(P_4)] *Let Riesz spaces X and Y be each Riesz isomorphic to an order ideal [a projection band] of the other space. Is X Riesz isomorphic to Y?*

We show that if X and Y have σ-complete Boolean algebras of projection bands then (P_4) has a positive answer (Theorem 3.4). Moreover, if X and Y are Banach lattices and the ideals in the assumption of (P_3) are norm closed then X^* and Y^* are Riesz isomorphic (Corollary 4.2), while X and Y may be not Riesz isomorphic (Example 4.3); similar theorems for L^1-preduals were also considered in [9], p. 229.

2. PRELIMINARIES

For terminology and notations concerning the general theory of Riesz spaces not explained below we refer to [1], [10] and [15]. In the present paper all Riesz spaces are assumed to be Archimedean. Let X be a Riesz space. For a nonempty subset A of X we define the disjoint complement A^d of A as $A^d=\{x\in X: |x|\wedge|y|=0 \text{ for all } y\in A\}$. A^d is a lattice ideal and $A^d\cap A^{dd}=0$. An ideal $B\subset X$ is said to be a [projection] band if $B=B^{dd}$ [and $X=B+B^{dd}$] or, equivalently, if it follows from $A\subset B$ and $x=\sup A$ that $x\in B$. The set $\mathscr{A}(X)$ of all bands in X, partially ordered by inclusion, is a complete Boolean algebra with operations $\vee$, $\wedge$ and a complementation ^{c} defined as follows: $B_1\vee B_2=(B_1+B_2)^{dd}$, $B_1\wedge B_2=B_1\cap B_2$ and $B^c=B^d$. The set $\mathscr{P}(X)$ of all projection bands in X is a Boolean subalgebra of $\mathscr{A}(X)$ with $\wedge$, ^{c} as in $A(X)$ and $B_1\vee B_2=B_1+B_2$. A lattice ideal $A\subset X$ is called order dense in X if $A^{dd}=X$. For every projection band B in X there exists a natural projection P_B from X onto B (the so called band projection). A projection P is a band projection iff $0\le Px\le x$ holds for every $x\ge 0$. The set $\mathscr{B}(X)$ of all band projections in X is a Boolean algebra with operations $\vee$, $\wedge$ and ^{c} defined as follows: $P_1\vee P_2=P_1+P_2-P_1P_2$, $P_1\wedge P_2=P_1P_2$ and $P^c=I-P$. The mapping $B\to P_B$ is a Boolean isomorphism from $\mathscr{P}(X)$ onto $\mathscr{B}(X)$; in particular $P_B^c=P_B^d$. Every band projection is an order continuous (a normal) Riesz homomorphism (i.e. $P(\sup_{t\in T}x_t)=\sup_{t\in T}P(x_t)$ holds for every index set T). Every positive operator (and hence every Riesz homomorphism) between F-lattices is norm continuous ([1], Th. 16.6, p. 111). We write $X\cong Y$ if X and Y are Riesz isomorphic. X^* denotes an order dual of X; if X is a normed space, then X' denotes the topological dual of X. X^* is Dedekind complete and if X is normed lattice, then X' is an ideal of X^* (and hence also Dedekind complete); if X is a Banach lattice, then $X'=X^*$ ([15], Prop. 5.5, p. 85). For a nonempty subset A of a Banach space X the symbol $A^\perp$ denotes the annihilator of A (in X').

A Riesz space X is said to have *the σ-property* if $\mathscr{P}(X)$ is σ-complete. If X is σ-Dedekind complete or has the projection property then X has the σ-property ([10], Th. 30.6, p. 177). In particular all order duals of Riesz spaces, all topological duals of normed lattices and all L^p-spaces $(1 \leq p < \infty)$ have the σ-property.

Let $\mathscr{B}$ be a Boolean algebra with operations $\vee$, $\wedge$ and ^{c}. For a fixed $a_0 \in \mathscr{B}$ the symbol $a_0 \wedge \mathscr{B}$ denotes the set $\{a_0 \wedge b : b \in \mathscr{B}\}$ (a principal ideal of $\mathscr{B}$). $a_0 \wedge \mathscr{B}$ is a Boolean algebra with operations $\vee$, $\wedge$ restricted from $\mathscr{B}$ and a complementation $'$ defined by the formula $a' = a_0 \wedge a^c$. The following fundamental theorem is due to R. Sikorski ([17], Th. 1 and Th. 2).

THEOREM 2.1. *Let $\mathscr{A}$, $\mathscr{B}$ be σ-complete Boolean algebras, $a_0 \in \mathscr{A}$, $b_0 \in \mathscr{B}$ and let F, G be boolean isomorphisms of $\mathscr{A}$ onto $b_0 \wedge \mathscr{B}$ and of $\mathscr{B}$ onto $a_0 \wedge \mathscr{A}$, respectively. Then there exist elements $a \in \mathscr{A}$ and $b \in \mathscr{B}$ such that*

$$F(a) = b \quad \textit{and} \quad G(b^c) = a^c.$$

3. MAIN RESULTS

The following lemma will be usefull.

LEMMA 3.1. *Let X and Y be Riesz spaces. If f is a Riesz isomorphism from X onto a band C_0 in Y, then the mapping F: $\mathscr{A}(X) \to \mathscr{A}(Y)$ defined by the formula $F(B) = f(B)$ is a Boolean isomorphism from $\mathscr{A}(X)$ onto $C_0 \wedge \mathscr{A}(Y)$. If moreover C_0 is a projection band, then F restricted to $\mathscr{P}(X)$ is onto $C_0 \wedge \mathscr{P}(Y)$.*

PROOF. Let for any nonempty subset A of C_0 the symbol A^D denotes the disjoint complement of A in C_0. It is easy to see that the mapping $F_1 : B \to f(B)$ is a Boolean isomorphism between $\mathscr{A}(X)$ and $\mathscr{A}(C_0)$, thus

(i) $F_1(B_1 \wedge B_2) = F_1(B_1) \wedge F_1(B_2)$, i.e. $f(B_1 \cap B_2) = f(B_1) \cap f(B_2)$;
(ii) $F_1(B_1 \vee B_2) = F_1(B_1) \vee F_1(B_2)$, i.e. $f(B_1 + B_2)^{dd}) = (f(B_1) + f(B_2))^{DD}$;
(iii) $F_1(B^d) = F_1(B)^D$, i.e. $f(B^d) = f(B)^D$.

We will show that $\mathscr{A}(C_0) = C_0 \wedge \mathscr{A}(Y)$. The inclusion $C_0 \wedge \mathscr{A}(Y) \subset \mathscr{A}(C_0)$ holds since the intersection of any band in Y with the ideal C_0 in Y is a band in C_0. Any band in C_0 is a band in Y (by (2) below, for example), thus $\mathscr{A}(C_0) = \{B \cap C_0 : B \in \mathscr{A}(C_0)\} \subset \{B \cap C_0 : B \in \mathscr{A}(Y)\} = C_0 \wedge \mathscr{A}(Y)$. Now let I be the identity mapping from $\mathscr{A}(C_0)$ onto $C_0 \wedge \mathscr{A}(Y)$. From the definition of a disjoint complement we have

(1) $\qquad A^D = C_0 \cap A^d$ for any nonempty subset A of C_0,

and by (1) and the distribution of $\mathscr{A}(Y)$ we get: $A^{DD} = (A^D)^d \cap C_0 = (A^d \wedge C_0)^d \wedge C_0 = (A^{dd} \vee C_0^d) \wedge C_0 = A^{dd}$ (since $A \subset C_0$ implies $A^{dd} \subset C_0$), so

(2) $\qquad A^{DD} = A^{dd}$ for any nonempty subset A of C_0.

From (1) and (2) we get $I(B_1 \wedge B_2) = I(B_1) \wedge I(B_2)$, $I(B_1 \vee B_2) = (B_1 + B_2)^{DD} = (B_1 + B_2)^{dd} = I(B_1) \vee I(B_2)$ and $I(B^D) = C_0 \wedge B^d$. Hence, I is a Boolean isomorphism and the mapping $F = IF_1$ is a Boolean isomorphism from $\mathscr{A}(X)$ onto $C_0 \wedge \mathscr{A}(Y)$.

To prove the second part of the lemma, observe that the mapping $P \rightarrow fPf^{-1}$ is a Boolean isomorphism between $\mathscr{B}(X)$ and $\mathscr{B}(C_0)$; therefore F_1 restricted to $\mathscr{P}(X)$ is a Boolean isomorphism onto $P(C_0)$. We will show that $\mathscr{P}(C_0) = C_0 \wedge \mathscr{P}(Y)$, i.e. I restricted to $\mathscr{P}(C_0)$ is onto $C_0 \wedge \mathscr{P}(Y)$. Since the restriction of any band projection in Y to the ideal C_0 is a band projection, then by using the natural isomorphism between $\mathscr{B}(Y)$ and $\mathscr{P}(Y)$ we have $C_0 \wedge \mathscr{P}(Y) = \{C_0 \cap B : B \in \mathscr{P}(Y)\} \subset \mathscr{P}(C_0)$. Now let P_0 be such a band projection in Y that $C_0 = P_0 Y$. It is easy to see that for every $P \in \mathscr{B}(C_0)$ the mapping PP_0 is an element of $\mathscr{B}(Y)$, hence $PC_0 = PP_0 Y$ is a projection band in Y. This implies $\mathscr{P}(C_0) \subset \mathscr{P}(Y)$ and we get $\mathscr{P}(C_0) \subset C_0 \wedge \mathscr{P}(Y)$.

PROPOSITION 3.2. *Let X and Y be Riesz spaces and A_0, B_0 bands in X, Y respectively. If $f : X \rightarrowtail\!\!\!\rightarrow B_0$ and $g : Y \rightarrowtail\!\!\!\rightarrow A_0$ are Riesz isomorphisms then there exist bands A and B in X and Y, respectively, such that the (order dense) ideals $I_A = A + A^d$ and $I_B = B + B^d$ are Riesz isomorphic. More precisely, the mapping h defined on I_A by the formula*

(3) $\qquad h_{|A^d} = f \quad \textit{and} \quad h_{|A^d} = g^{-1}$

is the required Riesz isomorphism.

PROOF. Let us define mappings F and G on $\mathscr{A}(X)$ and $\mathscr{A}(Y)$, respectively, as in Lemma 3.1. Since $\mathscr{A}(X)$ and $\mathscr{A}(Y)$ are complete, Lemma 3.1 and Theorem 2.1 imply that there exist elements $A \in \mathscr{A}(X)$ and $B \in \mathscr{A}(Y)$ such that

(4) $\qquad f(A) = B \quad \text{and} \quad g^{-1}(A^d) = B^d.$

It is easy to see that A and B are projection bands in I_A and I_B, respectively. Let Q denotes the corresponding to A band projection in I_A and observe that $Q^c(I_A) = A^d$. Put

(5) $\qquad h = fQ + g^{-1}Q^c;$

then h is a Riesz isomorphism from I_A into I_B. Moreover, by (3) we have $I_B = B + B^d = f(A) + g^{-1}(A^d) = (fQ)(I_A) + (g^{-1}Q^c)(I_A) = h(I_A)$, thus h is onto. Clearly (3) and (5) define the same Riesz isomorphism h.

REMARK 3.3. (i) In the above Proposition X and Y posess order ideals which are Riesz isomorphic. It will be proved that this isomorphism, in general, cannot be extended to a Riesz isomorphism from X onto Y (Example 4.3). Moreover, (ii) if X and Y have the projection property then they are Riesz isomorphic. The general case is considered in the below theorem.

THEOREM 3.4. ([22], Th. 3.2) *Let the Riesz spaces X and Y have the σ-property. If X and Y are each Riesz isomorphic to a projection band of the other space then they are Riesz isomorphic.*

PROOF. Let A_0 and B_0 be projection bands in X and Y, respectively, and f, g Riesz isomorphisms such that $f: X \rightarrowtail\!\!\!\twoheadrightarrow B_0$ and $g: Y \rightarrowtail\!\!\!\twoheadrightarrow A_0$. Since $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ are σ-complete, by the second part of Lemma 3.1 and an argument similar to that of 3.2 there exist projection bands $A \in \mathscr{P}(X)$ and $B \in \mathscr{P}(Y)$ for which (4) holds. Denote by P_A the band projection corresponding to A in X. It is easy to see that the mapping

(6) $$h = fP_A + g^{-1}P_A^c$$

is the required Riesz isomorphism from X onto Y.

REMARK 3.5. (i) If X and Y are F-lattices then all Riesz isomorphisms between them and all band projections in X and Y are norm continuous. In this case we get a lattice-topological version of (P_3). (ii) If X and Y are abstract M (or L^p)-spaces and the Riesz isomorphisms f and g are isometries, then h, defined both by (5) and (6), is an isometry as well.

4. COROLLARIES

COROLLARY 4.1. *Let the Riesz spaces X, Y have the projection property and let $f: X \twoheadrightarrow Y$, $g: Y \twoheadrightarrow X$ be Riesz homomorphisms. If f and g are order continuous (or, equivalently, $\ker f$ and $\ker g$ are (projection) bands) then X and Y are Riesz isomorphic.*

PROOF. X and Y are Riesz isomorphic to the projection bands $(\ker g)^d$ and $(\ker f)^d$, respectively. By Theorem 3.4 X and Y are Riesz isomorphic.

A Banach lattice $(X, \|\cdot\|)$ is said to have an order continuous norm if for every downwards directed set $\{x_\alpha\}_{\alpha \in A}$ in X with $\inf_{\alpha \in A} x_\alpha = 0$ $\lim_\alpha \|x_\alpha\| = 0$. Each such a Banach lattice is Dedekind complete and each norm closed ideal of X is a projection band ([15], Th. 5.10, p. 89 and Th. 5.14, p. 94).

COROLLARY 4.2. *Let X and Y be Banach lattices. If*

(a) *each of these spaces is Riesz isomorphic to a norm closed ideal of the other space, or*

(b) *$f: X \twoheadrightarrow Y$ and $g: Y \twoheadrightarrow X$ are Riesz homomorphisms,*

then X^* and Y^* are Riesz isomorphic. If moreover X and Y have order continuous norms, then X and Y are Riesz isomorphic too.

PROOF. The second part of the Corollary is a simple consequence of Theorem 3.4 and Corollary 4.1 ($\ker f$ and $\ker g$ are norm closed ideals in X and Y, respectively).

(a) Let f be a Riesz isomorphism from X onto a norm closed ideal of Y. The adjoint mapping $f^*: Y^* \twoheadrightarrow X^*$ is a Riesz homomorphism (see the remark after Prop. 5.6, p. 86 in [15]) and $\ker f^*$ is a projection band of Y^* ([15], Corollary 1, p. 86). Analoglously, if g is a Riesz isomorphism from Y onto a norm closed ideal of X then $\ker g^*$ is a projection band of X^*. By Corollary 4.1 X^* and Y^* are Riesz isomorphic.

(b) By ([15], Corollary 1, p. 86) we have $(X/_{\ker f})^* \cong (\ker f)^{\perp}$ is a projection band of X^* and since $Y \cong X/_{\ker f}$, we get $Y^* \cong (\ker f)^{\perp}$. Analoglously X^* is Riesz isomorphic to a projection band in Y^* and by Theorem 3.4 $X^* \cong Y^*$.

In [7] K. Kuratowski gave an example of uncountable and compact subsets Ω_1, Ω_2 of $\mathbb{R}^2$ such that

1°. *Ω_1 is homeomorphic to a closed-open subset A_0 of Ω_2 and Ω_2 is homeomorphic to a closed-open subset B_0 of Ω_1, but Ω_1 is not homeomorphic to Ω_2;*

2°. *The Boolean algebras $CO(\Omega_i)$ of closed-open subsets of Ω_i, $i = 1, 2$, are not σ-complete.*

Making use of this example we will show that the assumption concerning the σ-property in Theorem 3.4 is essential (even if f and g are isometries).

EXAMPLE 4.3. *There exist Banach lattices X and Y without the σ-property such that*

1. *X and Y are each Riesz isometric to a projection band of the other space, but $X \not\cong Y$;*
2. *X and Y posess order dense ideals which are Riesz isometric;*
3. *$X^* \cong Y^*$.*

PROOF. Each band projection in the Riesz space $C(\Omega)$ (of all real-valued continuous functions on a compact Hausdorff space Ω) is of the form $P_A x = x \cdot \chi_A$, where $x \in C(\Omega)$ and χ_A is the characteristic function of some closed-open subset $A \subset \Omega$ ([15], Ex. 5, p. 63), thus $\mathscr{P}(C(\Omega))$ is Boolean isomorphic to $CO(\Omega)$. Let us put $X = C(\Omega_1)$ and $Y = C(\Omega_2)$, where Ω_1, Ω_2 are Kuratowski's sets. By 2°. neither X nor Y have the σ-property. Now let f be the homeomorphism from Ω_1 onto the closed-open subset A_0 of Ω_2 described in 1°. The mapping $\bar{f}: C(A_0) \rightarrowtail\!\!\!\!\twoheadrightarrow X$ of the form $\bar{f}(x) = x \circ f$ is a Riesz isometry and the restriction $G(y) = y_{|A_0}$ is a Riesz isometry of $P_{A_0}(Y)$ onto $C(A_0)$, hence the mapping $F = (\bar{f} \circ G)^{-1}$ is a Riesz isometry of X onto the projection band $P_{A_0}(Y)$. Analoglously, Y is Riesz isometric to a projection band of X. The second part of 1. is a consequence of Corollary 1, p. 104 in [15], which states that if $C(\Omega_1) \cong C(\Omega_2)$ then Ω_1 and Ω_2 are homeomorphic.

Clearly Remark 3.5 (ii) and Corollary 4.2 (a) imply 2. and 3. respectively.

REMARK 4.4. (i) Notice that although $C(\Omega_1)$ and $C(\Omega_2)$ are not Riesz isomorphic, but by Milutin's Theorem ([16], Th. 12.5.10, p. 379) they are linearly homeomorphic (to $C(0, 1)$) because Ω_1 and Ω_2 are uncountable, compact and metric spaces. Other examples with the same properties as in 1°. and 2°. above can be found in [6], [18] and [19].

(ii) In 1971 K. Sundaresan ([18], see also [19], pp. 165–168) gave an example of a nonmetrizable, compact and Hausdorff space S such that $S \oplus p$ is not homeomorphic to S but $S \oplus p \oplus q$ is homeomorphic to S for each pair of distinct points $p, q \notin S$. Putting $X = C(S)$ and $Y = C(S \oplus p)$ we get a curious example of non-Riesz isomorphic Banach lattices such that $X \oplus \mathbb{R} \cong Y$, $Y \oplus \mathbb{R} \cong X$ and $X \oplus \mathbb{R} \not\cong X$ but $X \oplus \mathbb{R}^2 \cong X$. In view of (i) the following problem arises:

PROBLEM 4.5. *Is $C(S)$ linearly homemorphic to $C(S \oplus p)$?*

Now we will applicate Theorem 3.4 to some sequence spaces. If (x_i) is an unconditional Schauder basis in a real Banach space X, then X is a Dedekind complete Riesz space with an ordering $x \leq y$ iff $\forall i \in \mathbb{N}$ $x_i^*(x) \leq x_i^*(y)$. Moreover, each projection band in X is of the form $B = -\mathrm{in}\{x_i\}_{i \in \mathbb{N}_0}$, where $\mathbb{N}_0$ is some subset of $\mathbb{N}$ depending on B [1]. If Y is an another real Banach space with an unconditional basis (y_i) then the bases (x_i) and (y_i) are said to be [permutatively] equivalent if [there exists a permutation τ of $\mathbb{N}$ such that] $\sum_{i=1}^{\infty} \alpha_i x_i$ converges iff $\sum_{i=1}^{\infty} d_i y_i$ [$\sum_{i=1}^{\infty} \alpha_i y_{\tau(i)}$] converges for any real sequence (α_i). The [permutative] equivalence of (x_i) and (y_i) defines a Riesz isomorphism from X onto Y. The basic sequence (x_{i_k}) is called a subbasis of the basis (x_i). Applying Theorem 3.4 (with (6)) we get the following result.

COROLLARY 4.6. *Let X, Y be real Banach spaces with unconditional Schauder bases (x_i), (y_i) respectively. If (x_i) and (y_i) are each (permutatively) equivalent to a subbasis of the other basis then they are permutatively equivalent. In particular X and Y are linearly homeomorphic.*

REMARK 4.7. The above Corollary holds also for unconditional bases in F-spaces [21], but unfortunately this more general case cannot be obtained from Theorem 3.4 (such an F-space with the coordinatewise ordering is not a Riesz space - see [13], Th. 3.8.3, p. 155 and [14]).

An Orlicz function is a non-decreasing continuous function $f: R_+ \to R_+$ such that $f(0) = 0$ and $f \not\equiv 0$. Let $F = (f_n)$ be a sequence of Orlicz functions. For each $x = (x_n) \in \mathbb{R}^{\mathbb{N}}$ we define $m_F(x) = \sum_{i=1}^{\infty} \alpha_i f_n(|x_n|)$, the linear space $l_F = \{x \in \mathbb{R}^{\mathbb{N}} : m_F(tx) < \infty$ for some $t > 0\}$ and the F-norm on l_F as $\|x\|_F = \inf\{t > 0 : m_F(x/_t) \leq t\}$. Then $(l_F, \|\cdot\|_F)$ is called a modular sequence space. For the general theory of modular sequence spaces we refer the reader to [8] and [20]. l_F is a Dedekind complete F-lattice with pointwise ordering. Each projection band in l_F is of the form $B = \{x \in l_F : x_{|\mathbb{N}_0} \equiv 0\}$ and is Riesz isomorphic to $l_{(f_n)_{n \in \mathbb{N}_0}}$, where $\mathbb{N}_0$ is some subset of $\mathbb{N}$ depending on B. The sequences of Orlicz functions $F = (f_n)$ and $G = (g_n)$ are said to be [permutatively] equivalent if $l_F = l_G$ [$l_F = l_{(g_{\tau(n)})}$ for some permutation τ of $\mathbb{N}$]. The [permutative] equivalence of F and G defines a Riesz (and hence a topological) isomorphism from l_F onto l_G. Applying Theorem 3.4 we get

COROLLARY 4.8. *Let $F = (f_n)$ and $G = (g_n)$ be sequences of Orlicz functions. If F and G are each (permutatively) equivalent to a subsequence of the other sequence then they are permutatively equivalent. In particular l_F and l_G are linearly homeomorphic.*

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